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Curved Four-Dimensional Spacetime as Infrared Regulator in Superstring Theories

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ABSTRACT

We construct a new class of exact and stable superstring solutions in which our four-dimensional spacetime is taken to be curved . We derive in this space the full one-loop partition function in the presence of non-zero $\langle F^a_{\mu\nu}F^{\mu\nu}_a\rangle = F^2$ gauge background as well as in an $\langle R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\rangle = \mathcal{R}^2$ gravitational background and we show that the non-zero curvature, $Q^2 = 2/(k+2)$, of the spacetime provides an infrared regulator for all $\langle [F^a_{\mu\nu}]^n[R_{\mu\nu\rho\sigma}]^m\rangle$ correlation functions. The string one-loop partition function $Z(F,\mathcal{R},Q)$ can be exactly computed, and it is IR and UV finite. For Q small we have thus obtained an IR regularization, consistent with spacetime supersymmetry (when $F=0, \mathcal{R}=0$) and modular invariance. Thus, it can be used to determine, without any infrared ambiguities, the one-loop string radiative corrections on gravitational, gauge or Yukawa couplings necessary for the string superunification predictions at low energies.

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1 Introduction

The four-dimensional superstring solutions in a flat background [1]-[6] define at low energy effective supergravity theories [7],[8]. A class of them successfully extends the validity of the standard model up to the string scale, M_{string}. The first main property of superstrings is that they are ultraviolet-finite theories (at least perturbatively). Their second important property is that they unify gravity with all other interactions. This unification does not include only the gauge interactions, but also the Yukawa ones as well as the interactions among the scalars. This String Hyper Unification (SHU) happens at large energy scales $E_t \sim \mathcal{O}(M_{string}) \sim 10^{17}$ GeV. At this energy scale, however, the first excited string states become important and thus the whole effective low energy field theory picture breaks down[9, 10, 11, 12]. Indeed, the effective field theory of strings is valid only for $E_t \ll M_{string}$ by means of the $\mathcal{O}(E_t/M_{string})^2$ expansion. It is then necessary to evolve the SHU predictions to a lower scale $M_U < M_{string}$ where the effective field theory picture makes sense. Then, at M_U, any string solution provides non-trivial relations between the gauge and Yukawa couplings, which can be written as

$$\frac{k_i}{\alpha_i(M_U)} = \frac{k_j}{\alpha_j(M_U)} + \Delta_{ij}(M_U). \tag{1.1}$$

The above relation looks very similar to the well-known unification condition in Supersymmetric Grand Unified Theories (SuSy-GUTs) where the unification scale is about $M_U \sim 10^{16}$ GeV and $\Delta_{ij}(M_U) = 0$ in the $\bar{DR}$ renormalization scheme; in SuSy-GUTs the normalization constants k_i are fixed *only* for the gauge couplings ($k_1 = k_2 = k_3 = 1$, $k_{em} = \frac{3}{8}$), but there are no relations among gauge and Yukawa couplings at all. In string effective theories, however, the normalization constants (k_i) are known for both gauge and Yukawa interactions. Furthermore, $\Delta_{ij}(M_U)$ are calculable $finite$ quantities for any particular string solution. Thus, the predictability of a given string solution is extended for all low energy coupling constants $\alpha_i(M_Z)$ once the string-induced corrections $\Delta_{ij}(M_U)$ are determined.

This determination however, requests string computations which we did not know, up to now, how to perform in generality. It turns out that $\Delta_{ij}(M_U)$ are non-trivial functions of the vacuum expectation values of some gauge singlet fields [8], $\langle T_A \rangle = t_A$, the so-called moduli (the moduli fields are flat directions at the string classical level and they remain flat in string perturbation theory, in the exact supersymmetric limit) :

$$\Delta_{ij}(M_U) = E_{ij} + F_{ij}(t_A). \tag{1.2}$$

Here $F_{ij}(t_A)$ are modular forms, which depend on the particular string solution. Partial results for F_{ij} exist in the exact supersymmetric limit in many string solutions based on orbifold [2] and fermionic constructions [4]. The finite part E_{ij} is a function of M_U/M_{string} and, at the present time, it is only approximately estimated [8]. As we will see later E_{ij} are, in principle, well defined calculable quantities once we perform our calculations at the string level where all interactions including gravity are consistently defined. The full string corrections to the coupling constant unification, $\Delta_{ij}(M_U)$, as well as the string corrections associated to the soft supersymmetry-breaking parameters

$$m_0,\ m_{1/2},\ A,\ B \text{ and } \mu, \quad \text{at} \quad M_U, \tag{1.3}$$

are of main importance, since they fix the strength of the gauge and Yukawa interactions, the full spectrum of the supersymmetric particles as well as the Higgs and the top-quark masses at the low energy range $M_Z \leq E_t \leq \mathcal{O}(1)$ TeV.

In the case where supersymmetry is broken [13],[14] only semi-quantitative results can be obtained at present; a much more detailed study and understanding are necessary which is related to the structure of soft breaking terms after the assumed supersymmetry breaking [15].

The main obstruction in determining the exact form of the string radiative corrections $\Delta_{ij}(M_U)$ is strongly related to the infrared divergences of the $\langle [F^a_{\mu\nu}]^2 \rangle$ two-point correlation function in superstring theory. In field theory, we can avoid this problem using off-shell calculations. In first quantized string theory we cannot do that since we do not know how to go off-shell. Even in field theory there are problems in defining an infrared regulator for chiral fermions especially in the presence of spacetime supersymmetry.

In [16] it was suggested to use a specific spacetime with negative curvature in order to achieve consistent regularization in the infrared. The proposed curved space however is not useful for string applications since it does not correspond to an exact super-string solution.

Recently, exact and stable superstring solutions have been constructed using special four-dimensional spaces as superconformal building blocks with $\hat{c} = 4$ and $N = 4$ superconformal symmetry [9], [11]. The full spectrum of string excitations for the superstring solutions based on those four-dimensional subspaces, can be derived using the techniques developed in ref. [11]. The main characteristic property of these solutions is the existence of a mass gap $\mu^2 = Q^2/4$, which is proportional to the curvature of the non-trivial four-dimensional spacetime. Comparing the spectrum in a flat background with that in curved space we observe a shifting of all massless states by an amount proportional to the spacetime curvature, $\Delta m^2 = Q^2/4$. What is also interesting is that the shifted spectrum in the curved space is equal for bosons and fermions due to the existence of a new space-time supersymmetry defined in the curved spacetime [9] [11]. Therefore, our curved space time infrared regularization (CSIR) is consistent with supersymmetry and can be used either in field theory or string theory.

In section 2 we define the four-dimensional superconformal system and give the modular-invariant partition function for some symmetric orbifold ground states of the string. In section 3 we show that we can deform the theory consistently, by switching on a non-zero gauge field strength background $\langle F^a_{\mu\nu} F^{\mu\nu}_a \rangle = F^2$ or a gravitational one, $\langle R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \rangle = \mathcal{R}^2$ and obtain the exact regularized partition function $Z(Q, F, \mathcal{R})$. Our method of constructing this effective action automatically takes into account the back-reaction of the other background fields; stated otherwise, the perturbation that turns on the constant gauge field strength or curvature background is an exact (1,1) integrable perturbation. The second derivative with respect to r of our deformed partition function $\partial^2 Z(Q,F,\mathcal{R})/\partial F^2$ for $F,\mathcal{R}=0$ defines without any infrared ambiguities the complete string one-loop corrections to the gauge coupling constants. In the $Q \to 0$ limit we recover the known partial results [8].

2 Superstrings in Curved Space Time

As usual in order to construct a four-dimensional superstring solution one must saturate the superconformal anomaly $\hat{c}=10$ combining two sub-systems.

i) the four-dimensional space time superconformal system with $\hat{c}=4+\epsilon$ and

ii) a six-dimensional compact space with $\hat{c}=6-\epsilon$.

In all our constructions we impose $\epsilon=0$ so that the internal compact space can be chosen to be in one-to-one correspondence with any possible construction in four-dimensional flat space. The idea here is to replace the four free space time supercoordinates $\hat{c}=4$ with a non-trivial $\hat{c}=4$ Euclidean spacetime which shares similar superconformal properties, namely an $N=4$ superconformal symmetry. In ref. [18] a large class of such spaces is found. Although all these solutions are exact superconformal systems due to the $N=4$ symmetry, we will restrict in what follows to the four $N=4$ realizations of ref.[19] and [9] which are based on (gauged-) Wess-Zumino-Witten models mainly because we know well their characters and thus can construct explicitly the one-loop partition of the full string model.

As we already mentioned above, all four-dimensional superstring solutions are of the form $F^{(4)} \otimes K^{(6)}$, while the curved four-dimensional superstring solutions replace the four-dimensional flat space time ($F^{(4)}$) with one of the following possibilities [9]:

1) $W_k^{(4)} \equiv U(1)_Q \otimes SU(2)_{k_1}$

2) $C_k^{(4)} \equiv [SU(2)/U(1)]_k \otimes U(1)_R \otimes U(1)_Q$

3) $\Delta_k^{(4)}(A) \equiv [SU(2)/U(1)]_k \otimes [SL(2,R)/U(1)_A]_{k+4}$

4) $\Delta_k^{(4)}(V) \equiv [SU(2)/U(1)]_k \otimes [SL(2,R)/U(1)_V]_{k+4}$

The background Q in cases 1) and 2) is related to the level k due to the $N=4$ algebra, $Q=\sqrt{2/(k+2)}$ and guarantees that $\hat{c}=4$ for any value of k.

In the limit of weak curvature (large k) the $W_k^{(4)}$ space can be interpreted as a topologically non-trivial four-dimensional manifold of the form $R \otimes S^3$. The underlying superconformal field theory associated to $W_k^{(4)}$ includes a supersymmetric $SU(2)_k$ WZW model describing the three coordinates of S^3 as well as a non-compact dimension with a background charge, describing the scale factor of the sphere [9],[11]. Furthermore this space admits two covariantly constant spinors and, therefore, respects up to two space-time supersymmetries consistently with the $N=4$ world-sheet symmetry [20, 9, 11]. The explicit representation of the desired $N=4$ algebra is derived in [19] and [9], while the target space interpretation as a four-dimensional semi-wormhole space is given in [20].

The space $C_k^{(4)}$ is factorized in two 2-d subspaces; for small curvatures the first subspace is described by the $SU(2)/U(1)$ bell, while the second subspace $U(1)_R \otimes U(1)_Q$ defines a two-dimensional cylinder. On the other hand, the spaces $W_k^{(4)}$ and $C_k^{(4)}$ are related to each other by target space duality transformation and both share the $N=4$ superconformal properties. The explicit realization of the $C_k^{(4)}$ space is given in [9]. From the conformal theory viewpoint $C_k^{(4)}$ is based on the supersymmetric gauged WZW model $C_k^{(4)} \equiv [SU(2)/U(1)]_k \otimes U(1)_R \otimes U(1)_Q$ with a background charge $Q=\sqrt{2/(k+2)}$ in one of the two coordinate currents $U(1)_Q$. The other free coordinate $U(1)_R$ is compactified on a torus with radius $R=\sqrt{k}$.

The $\Delta_k^{(4)}(A,V)$ spaces [9],[11] are also factorized in two 2-d subspaces; the first one is the $[SU(2)/U(1)]$ bell, while the second one is described by either the $[SL(2,R)/U(1)_A]$ cigar(axial gauging) or the $[SL(2,R)/U(1)_V]$ trumpet (vector gauging). In the $\Delta_k^{(4)}(A,V)$ spaces the elementary fields are the $[SU(2)/U(1)]_k$ (compact) parafermionic currents as well as the $[SL(2,R)/U(1)]_{k'}$ non-compact parafermionic currents. The level $k'=k+4$, so that the total central charge $\hat{c}$ remains equal to 4 for any value of k.

2.1 The $W_k^{(4)} \otimes K^{(6)}$ partition function

The basic rules of construction in curved space time are similar to that of the orbifold construction [2], the free 2-d fermionic constructions [4], and the Gepner construction [6] where one combines in a modular-invariant way the world-sheet degrees of freedom consistently with unitarity and spin-statistics of the string spectrum. We will choose as a first example the derivation of the string spectrum in background $W_k^{(4)} \otimes K^{(6)}$, where $K^{(6)}$ six-dimensional space. For definiteness we choose this space to be one of the symmetric orbifold model, used in $(2,2)$ compactifications.

Since the world-sheet fermions of the $W_k^{(4)}$ superconformal system are free and since the $K^{(6)}$ internal space is the same as in the $F^{(4)} \otimes K^{(6)}$ superstring solutions, we can easily obtain the partition function of $W_k^{(4)} \otimes K^{(6)}$, for k even, in terms of that of $F^{(4)} \otimes K^{(6)}$:

$$Z_W[Q,\tau,\bar{\tau}] = [\Gamma(SU(2)_k)(\tau,\bar{\tau})]\ Z^F[\tau,\bar{\tau}], \tag{2.4}$$

where $\Gamma(SU(2)_k)$ is nothing but the contribution to the partition function of the bosonic coordinates X^μ of the curved background $W^{(4)}$ divided by the contribution of the four free coordinates of the $F^{(4)}$ flat space,

$$\Gamma(SU(2)_k) = \frac{1}{2}[(\mathrm{Im}\tau)^{\frac{1}{2}}\eta(\tau)\bar{\eta}(\bar{\tau})]^3 \quad \sum_{a,b=0}^{1} Z^{SU(2)}[^a_b]. \tag{2.5}$$

$$Z^{SU(2)}[^a_b] = e^{-i\pi kab/2} \sum_{l=0}^{k} e^{i\pi bkl/2} \chi_l(\tau)\bar{\chi}_{l+a(k-2l)}(\bar{\tau}) \tag{2.6}$$

where $\chi_l(\tau)$ are the characters of $SU(2)_k$ (see for example [21]) and the integer l is equal to twice the $SU(2)$ spin $l = 2j$. It is necessary to use this orbifoldized version of $SU(2)_k$ comes in order to project out negative norm states of the $N = 4$ superconformal representations, [11].

To obtain the above formula we have used the continuous series of unitary representations of the Liouville characters [11] which are generated by the lowest-weight operators,

$$e^{\beta X_L} \ ; \quad \beta = -\frac{1}{2}Q + ip \ , \tag{2.7}$$

having positive conformal weights $h_p = Q^2/8 + p^2/2$. The fixed imaginary part in the momentum $iQ/2$ of the plane waves is due to the non-trivial dilaton motion.

As a particular example we give below the partition function of the $Z^2 \otimes Z^2$ symmetric orbifolds[2], [4], $W_k^{(4)} \otimes T^{(6)}/(Z^2 \otimes Z^2)$, for type-II and heterotic constructions:

$$\begin{aligned} Z_{II}^W[Q;\tau,\bar\tau] \ &= \ \frac{\Gamma(SU(2)_k)}{\mathrm{Im}\tau \ \eta^2\bar\eta^2} \times \frac{1}{16} \sum_{\alpha,\beta,\bar\alpha,\bar\beta=0}^{1} \ \sum_{h_1,g_1,h_2,g_2} Z_1\big[{}^{h_1}_{g_1}\big] Z_2\big[{}^{h_2}_{g_2}\big] Z_3\big[{}^{-h_1-h_2}_{-g_1-g_2}\big] \times \\ (-)^{\alpha+\beta+\alpha\beta} & \frac{\vartheta[{}^{\alpha}_{\beta}]}{\eta} \frac{\vartheta[{}^{\alpha+h_1}_{\beta+g_1}]}{\eta} \frac{\vartheta[{}^{\alpha+h_2}_{\beta+g_2}]}{\eta} \frac{\vartheta[{}^{\alpha-h_1-h_2}_{\beta-g_1-g_2}]}{\eta} \times (-)^{\bar\alpha+\bar\beta+\bar\alpha\bar\beta} \frac{\bar\vartheta[{}^{\bar\alpha}_{\bar\beta}]}{\bar\eta} \frac{\bar\vartheta[{}^{\bar\alpha+h_1}_{\bar\beta+g_1}]}{\bar\eta} \frac{\bar\vartheta[{}^{\bar\alpha+h_2}_{\bar\beta+g_2}]}{\bar\eta} \frac{\bar\vartheta[{}^{\bar\alpha-h_1-h_2}_{\bar\beta-g_1-g_2}]}{\bar\eta} \end{aligned} \tag{2.8}$$

where $Z_i[{}^{h_i}_{g_i}]$ in (2.8) stands for the partition function of two twisted bosons with twists (h_i, g_i). The untwisted part $Z_i[{}^0_0]$ is equal to the moduli-dependent two-dimensional lattice $\Gamma(2,2)[T_i,U_i]/(\eta\bar\eta)^2$. The definition of the ϑ-function we use is

$$\vartheta[{}^a_b](v|\tau) = \sum_{n\in Z} e^{i\pi\tau(n+a/2)^2+2i\pi(n+a/2)(v+b/2)} \tag{2.9}$$

In the heterotic case, a modular-invariant partition function can be easily obtained using the heterotic map [5], [6]. It consists in replacing in (2.8) the $O(2)$ characters associated to the right-moving fermionic coordinates $\bar\Psi^\mu$, with the characters of either $O(10) \otimes E_8$:

$$(-)^{\bar\alpha+\bar\beta+\bar\alpha\bar\beta} \frac{\bar\vartheta[{}^{\bar\alpha}_{\bar\beta}]}{\bar\eta} \to \frac{\bar\vartheta[{}^{\bar\alpha}_{\bar\beta}]^5}{\bar\eta^5} \frac{1}{2} \sum_{\gamma,\delta} \frac{\bar\vartheta[{}^{\gamma}_{\delta}]^8}{\bar\eta^8} \tag{2.10}$$

or $O(26)$:

$$(-)^{\bar\alpha+\bar\beta+\bar\alpha\bar\beta} \frac{\bar\vartheta[{}^{\bar\alpha}_{\bar\beta}]}{\bar\eta} \to \frac{\bar\vartheta[{}^{\bar\alpha}_{\bar\beta}]^{13}}{\bar\eta^{13}}. \tag{2.11}$$

Using the map above, the heterotic partition function with $E_8 \otimes E_6$ unbroken gauge group is:

$$\begin{aligned} Z_{het}^W[Q;\tau,\bar\tau] &= \frac{\Gamma(SU(2)_k)}{\mathrm{Im}\tau \ \eta^2\bar\eta^2} \times \frac{1}{16} \sum_{\alpha,\beta,\bar\alpha,\bar\beta=0}^{1} \ \sum_{h_1,g_1,h_2,g_2} Z_1\big[{}^{h_1}_{g_1}\big] Z_2\big[{}^{h_2}_{g_2}\big] Z_3\big[{}^{-h_1-h_2}_{-g_1-g_2}\big] \times \\ (-)^{\alpha+\beta+\alpha\beta} & \frac{\vartheta[{}^{\alpha}_{\beta}]}{\eta} \frac{\vartheta[{}^{\alpha+h_1}_{\beta+g_1}]}{\eta} \frac{\vartheta[{}^{\alpha+h_2}_{\beta+g_2}]}{\eta} \frac{\vartheta[{}^{\alpha-h_1-h_2}_{\beta-g_1-g_2}]}{\eta} \times \frac{1}{2} \sum_{\gamma,\delta} \frac{\bar\vartheta[{}^{\gamma}_{\delta}]^8}{\bar\eta^8} \frac{\bar\vartheta[{}^{\bar\alpha}_{\bar\beta}]}{\bar\eta^5} \frac{\bar\vartheta[{}^{\bar\alpha+h_1}_{\bar\beta+g_1}]}{\bar\eta} \frac{\bar\vartheta[{}^{\bar\alpha+h_2}_{\bar\beta+g_2}]}{\bar\eta} \frac{\bar\vartheta[{}^{\bar\alpha-h_1-h_2}_{\bar\beta-g_1-g_2}]}{\bar\eta} \end{aligned} \tag{2.12}$$

The mass spectrum of bosons and fermions in both the heterotic and type-II constructions is degenerate due to the existence of space-time supersymmetry defined in the $W_k^{(4)}$ background. The heterotic constructions are $N = 1$ spacetime supersymmetric while in the type-II construction one obtains $N = 2$ supersymmetric solutions.

The boson (or fermion) spectrum is obtained by setting to $+1$ (or to -1) the statistical factor, $(-)^{\alpha+\beta+\bar{\alpha}+\bar{\beta}+\alpha\beta+\bar{\alpha}\bar{\beta}}$, in the type-II construction, while one must set the statistical factor $(-)^{\alpha+\beta}{=}{+1}$ (or -1) in the heterotic constructions. In order to derive the lower-mass levels we need the behaviour of the bosonic and fermionic part of the partition function in the limit where $\mathrm{Im}\tau$ is large ($\mathrm{Im}\tau \to \infty$). This behaviour can be easily derived from the above partition functions.

$$Z^W(Q;\tau,\bar{\tau}) \longrightarrow \mathrm{C}[\mathrm{Im}\tau]^{-1}\ e^{-\frac{\mathrm{Im}\tau}{2(k+2)}}. \tag{2.13}$$

The above behaviour is universal and does not depend on the choice of K^6 internal $N = (2, 2)$ space. Only the multiplicity factor C (positive for bosons and negative for fermions) depends on the different constructions and it is always proportional to the number of the lower-mass level states $\mu^2 = 1/[2(k + 2)] = Q^2/4$. If we replace the $W_k^{(4)}$ with any one of the other $N = 4$ $\hat{c} = 4$ spaces, $C_k^{(4)}$,$\Delta_k^{(4)}(A, V)$, we get identical infrared mass shifting μ.

As we will see in the next section, the induced mass μ acts as a well-defined infrared regulator for all the on-shell correlation functions and in particular for the two-point function correlator $\langle F^a_{\mu\nu} F_a^{\mu\nu}\rangle$ (and $\langle R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}\rangle$) on the torus, which is associated to the one-loop string corrections on the gauge coupling constant.

3 Non-zero $\mathbf{F}^{\mathbf{a}}_{\mu\nu}$ and $R^{\rho\sigma}_{\mu\nu}$ Background in Superstrings

Our aim is to define the deformation of the two-dimensional superconformal theory which corresponds to a non-zero field strength $F^a_{\mu\nu}$ background and find the integrated one-loop partition function $\mathbf{Z}^W(Q, F, \mathcal{R})$, where F is by the magnitude of the field strength, $F^2 \equiv \langle F^a_{\mu\nu} F_a^{\mu\nu}\rangle$ and $\mathcal{R}$ is that of the curvature, $\langle R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}\rangle = \mathcal{R}^2$.

$$\mathbf{Z}^W[Q, F, \mathcal{R}] = \frac{1}{V(W)} \int_{\mathcal{F}} \frac{d\tau d\bar{\tau}}{(\mathrm{Im}\tau)^2} Z^W[Q, F, \mathcal{R}; \tau, \bar{\tau}] \tag{3.14}$$

where $V(W)$ is the volume of the $W_k^{(4)}$ space; modulo the trivial infinity which corresponds to the one non-compact dimension, the remaining three-dimensional compact space is that of the three-dimensional sphere. In our normalization:

$$V(SU(2)_k) = \frac{1}{8\pi}(k + 2)^{\frac{3}{2}}$$

so that it matches in the flat limit with the conventional flat space contribution.

In flat space, a small non-zero $F^a_{\mu\nu}$ background gives rise to an infinitesimal deformation of the 2-d σ-model action given by,

$$\Delta S^{2d}(F^{(4)}) = \int dz d\bar{z} F^a_{\mu\nu}[x^\mu \partial_z x^\nu + \psi^\mu \psi^\nu]\bar{J}_a \tag{3.15}$$

Observe that for $F^a_{\mu\nu}$ constant (constant magnetic field), the left moving operator $[x^\mu \partial_z x^\nu + \psi^\mu \psi^\nu]$ is not a well-defined $(1,0)$ operator on the world sheet. Even though the right moving Kac-Moody current $\bar{J}_a$ is a well-defined $(0,1)$ operator, the total deformation is not integrable in flat space. Indeed, the 2-d σ-model β-functions are not satisfied in the presence of a constant magnetic field. This follows from the fact that there is a *non-trivial back-reaction* on the gravitational background due the non-zero magnetic field.

The important property of $W^{(4)}_k$ space is that we can solve this back-reaction ambiguity. First observe that the deformation that corresponds to a constant magnetic field $B^a_i = \epsilon_{oijk} F^{ik}_a$ is a well-defined (1,1) integrable deformation, which breaks the $(2,2)$ superconformal invariance but keeps the $(1,0)$:

$$\Delta S^{2d}(W^{(4)}_k) = \int dz d\bar{z} B^a_i [I^i + \frac{1}{2}\epsilon^{ijk}\psi_j \psi_k]\bar{J}_a \tag{3.16}$$

where I^i is anyone of the $SU(2)_k$ currents. The deformed partition function is not zero due to the breaking of $(2,2)$ supersymmetry. In order to see that this is the correct replacement of the Lorentz current in the flat case, we will write the SU(2) group element as $g = \exp[i\vec{\sigma}\cdot\vec{x}/2]$ in which case $I^i = kTr[\sigma^i g^{-1}\partial g] = ik(\partial x^i + \epsilon^{ijk}x_j \partial x_k + \mathcal{O}(|x|^3))$. In the flat limit the first term corresponds to a constant gauge field and thus pure gauge so the only relevant term is the second one that corresponds to constant magnetic field in flat space. The $\mathcal{R}$ perturbation is

$$\Delta S(\mathcal{R}) = \int dz d\bar{z} \mathcal{R}\left[I^3 + \psi^1\psi^2\right]\bar{I}^3 \tag{3.17}$$

In σ-model language, in the flat limit it gives a metric perturbation

$$\delta(ds^2) = -\mathcal{R}\left[x^1 dx^2 - x^2 dx^1\right]^2 \tag{3.18}$$

with constant Riemann tensor and scalar curvature equal to $6\mathcal{R}$. There is also a non-zero antisymetric tensor with $H_{123} = 2\sqrt{\mathcal{R}}$ and dilaton $\delta\Phi = \mathcal{R}\left[(x^1)^2 + (x^2)^2 + 4(x^3)^2\right]/4$.

Due to the rotation invariance in S^3 we can choose $B^a_i = F\delta^3_i$ without loss of generality . The vector r_a indicates the direction in the gauge group space of the right-moving affine currents. Looking at the σ-model representation of this perturbation, we can observe that the $F_{\mu\nu}$ of this background gauge field is a monopole-like gauge field on S^3 and its lift to the tangent space is constant. Thus at the flat limit of the sphere it goes to the constant $F_{\mu\nu}$ background of flat space.

The moduli space of the $F, \mathcal{R}$ deformation is then given by the $SO(1,n)/SO(n)$ Lorentzian-lattice boostings with n being the rank of the right-moving gauge group. We therefore conclude that the desired partition function $\mathbf{Z}^W(Q, F, \mathcal{R})$ is given in terms of the moduli of the $\Gamma(1,n)$ lorentzian lattice. The constant gravitational background

$R^{ij}_{kl} = \mathcal{R}\epsilon^{3ij}\epsilon_{3kl}$ can also be included exactly by an extra boost, in which case the lattice becomes $\Gamma(1, n+1)$.

Let us denote by Q the fermionic lattice momenta associated to the left-moving $U(1)$ current $\partial H = \psi^1\psi^2$, by I the charge lattice of the left-moving $U(1)$ current associated to the I_3 current of $SU(2)_k$, by $\bar{Q}$ the charge lattice of a right U(1) which is part of the Cartan algebra of the non-abelian right gauge group and by $\bar{I}$ the charge lattice of the right-moving $U(1)$ current associated to the $\bar{I}_3$ current of $SU(2)_k$. In terms of these charges the part of the undeformed partition function that depends on them can be written as

$$Tr[\exp[-2\pi\mathrm{Im}\tau(\mathrm{L}_0 + \bar{\mathrm{L}}_0) + 2\pi\mathrm{i}\mathrm{Re}\tau(\mathrm{L}_0 - \bar{\mathrm{L}}_0)]] \tag{3.19}$$

where

$$L_0 = \frac{1}{2}Q^2 + \frac{I^2}{k}\ ,\ \bar{L}_0 = \frac{1}{2}\bar{Q}^2 + \frac{\bar{I}^2}{k} \tag{3.20}$$

The (1,1) perturbation that turns on a constant gauge field strength F as well as a constant curvature $\mathcal{R}$ background produces an special 2-parameter $O(2,2)$ boost in the charge lattice above, which transforms L_0 and $\bar{L}_0$ to

$$L_0' = L_0 + \frac{\cosh\psi - 1}{2}\left(\frac{(Q+I)^2}{k+2} + \left(\cos\theta\frac{\bar{I}}{\sqrt{k}} + \sin\theta\frac{\bar{Q}}{\sqrt{2}}\right)^2\right) + \tag{3.21}$$

$$+\sinh\psi\frac{(Q+I)}{\sqrt{k+2}}\left(\cos\theta\frac{\bar{I}}{\sqrt{k}} + \sin\theta\frac{\bar{Q}}{\sqrt{2}}\right)$$

and

$$L_0' - \bar{L}_0' = L_0 - \bar{L}_0 \tag{3.22}$$

The parameters θ and ψ are related to the constant background fields F and $\mathcal{R}$ by*

$$F = \frac{\sinh\psi\sin\theta}{\sqrt{2(k+2)}}\ \ ,\ \ \mathcal{R} = \frac{\sinh\psi\cos\theta}{\sqrt{k(k+2)}} \tag{3.23}$$

so that

$$L_0' - L_0 = (Q+I)\left(\mathcal{R}\bar{I} + F\bar{Q}\right) + \tag{3.24}$$

$$+\frac{\sqrt{1+(k+2)(2F^2+k\mathcal{R}^2)}-1}{2}\left(\frac{(Q+I)^2}{k+2} + \frac{\left(\mathcal{R}\bar{I} + +F\bar{Q}\right)^2}{(2F^2+k\mathcal{R}^2)}\right)$$

The first term is the standard perturbation while the second term is the back-reaction necessary for conformal and modular invariance. Expanding the partition function in a power series in $F, \mathcal{R}$

$$Z^W(Q, F, \mathcal{R}) = \sum_{n,m=0}^{\infty}\frac{F^n\mathcal{R}^m}{n!m!}Z^W_{n,m}(Q) \tag{3.25}$$

*The k-dependence is such that there is smooth flat space limit.

we can obtain the integrated correlators $\langle F^n R^m\rangle$. For $\langle F^2\rangle$, $\langle FR\rangle$ and $\langle R^2\rangle$ we obtain:

$$Z^W_{2,0}(Q) = -2\pi \mathrm{Im}\tau \left[2(Q+I)^2 + (k+2)\bar{Q}^2 - 8\pi\mathrm{Im}\tau(Q+I)^2\bar{Q}^2\right] \tag{3.26}$$

$$Z^W_{1,1}(Q) = -2\pi \mathrm{Im}\tau \left[k+2 - 8\pi\mathrm{Im}\tau(Q+I)^2\right]\bar{Q}\bar{I} \tag{3.27}$$

$$Z^W_{0,2}(Q) = -2\pi \mathrm{Im}\tau \left[k(Q+I)^2 + (k+2)\bar{I}^2 - 8\pi\mathrm{Im}\tau(\mathrm{Q}+\mathrm{I})^2\bar{\mathrm{I}}^2\right] \tag{3.28}$$

The charges Q_i in the above formula act in the respective $\vartheta\left[{\alpha \atop \beta}\right](\tau,v)$-functions as differentiation with respect to v. In particular Q acts in the $\vartheta[{\alpha \atop \beta}]$ in equ. (7), $I, \bar{I}$ act in the level-k ϑ-function present in $\Gamma(SU(2)_k)$ (due to the parafermionic decomposition), and $\bar{Q}$ acts on one of the right $\bar{\vartheta}$-functions.

We are interested in the one-loop correction to the gauge couplings, which is proportional to $Z^W_{2,0}(Q)$. We can use the Riemann identity to transform the sum over the (α,β) ϑ-function characteristics (with non-zero v) that appear in (7,10) into

$$\frac{1}{2}\sum_{a,b=0}^{1}(-)^{\alpha+\beta+\alpha\beta}\vartheta[{\alpha \atop \beta}](v|\tau)\vartheta[{\alpha+h_1 \atop \beta+g_1}](0|\tau)\vartheta[{\alpha+h_2 \atop \beta+g_2}](0|\tau)\vartheta[{\alpha-h_1-h_2 \atop \beta-g_1-g_2}](0|\tau) = \tag{3.29}$$

$$= \vartheta[{1 \atop 1}](v/2|\tau)\vartheta[{1-h_1 \atop 1-g_1}](v/2|\tau)\vartheta[{1-h_2 \atop 1-g_2}](v/2|\tau)\vartheta[{1+h_1+h_2 \atop 1+g_1+g_2}](v/2|\tau)$$

In this representation the charge operators are derivatives with respect to v.

We will focus for simplicity to heterotic $Z_2\times Z_2$ orbifolds. In this case all the characteristics in eq. (2.12) take the values $0,1$. The only non-zero contribution appears when one of the pairs (h_i,g_i) of twists is $(0,0)$ and the rest non-zero. There are three sectors where two out of the four fermion ϑ-functions depend only on $v/2$; they give non-zero contribution only when both derivatives with respect to v act on them. We have in total three $N=2$ sectors; the $N=4$ and the $N=1$ sectors give zero contribution in $Z_{2,0}(Q)$ for the $Z^2\otimes Z^2$ orbifold model. For other orbifold models there will be non-zero contributions from the N=1 sectors. Using the fact that the contribution to the partition function of the twisted bosons cancels (up to a constant that is proportional to the number of fixed points) that of the twisted fermions, and also the identity $\vartheta'(0)/2\pi=\eta^3$, we obtain the following formula for $Z_{2,0}(Q)$:

$$Z^A_{2,0}(Q) = -\sum_{i=1}^{3}\int_{\mathcal{F}}\frac{d\tau d\bar{\tau}}{\mathrm{Im}\tau}\frac{\Gamma(SU(2))}{V(SU(2))}\frac{\Gamma^i_{2,2}(T_i,U_i)}{\bar{\eta}^{24}}\left[\bar{Q}_A^2 - \frac{1}{4\pi\mathrm{Im}\tau}\right]2\bar{\Omega}(\bar{\tau}) \tag{3.30}$$

where A indicates the appropriate gauge group (E_8,E_6 or $U(1)$), $\bar{Q}_A$ is the associated charge operator, normalized so that it acts as $\frac{i}{\pi}\frac{\partial}{\partial\bar{\tau}}$ on the ϑ-functions and $\bar{\Omega}=\bar{\Omega}_8\bar{\Omega}_6$ with

$$\bar{\Omega}_8(\bar{\tau}) = \frac{1}{2}\sum_{a,b=0}^{1}\bar{\vartheta}^8[{a \atop b}]\ ,\ \ \bar{\Omega}_6(\bar{\tau}) = \frac{1}{4}\left[\bar{\vartheta}_2^8(\bar{\vartheta}_3^4+\bar{\vartheta}_4^4) - \bar{\vartheta}_4^8(\bar{\vartheta}_3^4+\bar{\vartheta}_2^4) + \bar{\vartheta}_3^8(\bar{\vartheta}_2^4-\bar{\vartheta}_4^4)\right] \tag{3.31}$$

Thus the one-loop corrected gauge coupling constant can be written as

$$\frac{16\pi^2}{g_A^2(Q)} = \frac{16\pi^2}{g_A^2(M_{str})} + Z_{2,0}^A \tag{3.32}$$

Eq. (3.30) applies to any 4-d symmetric orbifold string model, the only things that change are the moduli contribution Γ^i and the specific form of $\bar{\Omega}$. This formula differs from that of [8] since it includes the so-called universal contribution and is UV and IR *finite*. In particular the back-reaction of gravity is included exactly and contributes to the universal terms. Taking differences between different gauge groups we obtain the regularized form of the result of [8]. The only difference from their formula is the replacement of the flat space contribution by $\Gamma(SU(2))/V(SU(2))$. Our result is *explicitly modular invariant* and *finite*.

In order to clearly see how the $W_k^{(4)}$ acts as an IR regulator, it is convenient to perform the summation on the spin index l of the $SU(2)$ characters. This sum can be done analytically and one obtains the following surprising (and eventually useful) identity

$$\Gamma(SU(2)_k) = \sqrt{\mathrm{Im}\tau}\frac{(k+2)^{3/2}}{4\pi}\left[\frac{\partial Z(R)}{\partial R}|_{R^2=k+2} - \frac{1}{2}\frac{\partial Z(R)}{\partial R}|_{R^2=(k+2)/4}\right] \tag{3.33}$$

where $Z(R)$ is the $\Gamma(1,1)$ lattice contribution of the torus:

$$Z(R) = \sum_{m,n} \exp\left[\frac{i\pi\tau}{2}(\frac{m}{R}+nR)^2 - \frac{i\pi\bar{\tau}}{2}(\frac{m}{R}-nR)^2\right] \tag{3.34}$$

Using the identity above, $Z_2(Q)$ becomes,

$$Z_2^A(Q) = \sum_{i=1}^{3} 2\int_{\mathcal{F}} \frac{d\tau d\bar{\tau}}{\mathrm{Im}\tau^2}\mathrm{Im}\tau^{\frac{1}{2}}\left[Z'(R)|_{k+2} - \frac{1}{2}Z'(R)|_{(k+2)/4}\right]\left[\mathrm{Im}\tau\Gamma^i_{(2,2)}(T_i,U_i)\Sigma^A\right] \tag{3.35}$$

The function Σ^A depends on the gauge group in question and its constant part is $C(g_A) - T(R_A^i)$. For example, in the E_8 case it is given by

$$\Sigma^{E_8} = -2\frac{\bar{\Omega}_6}{\bar{\eta}^{24}}\left[\frac{i\partial}{\pi\partial\bar{\tau}} - \frac{1}{4\pi\mathrm{Im}\tau}\right]\bar{\Omega}_8 \tag{3.36}$$

The combination $\frac{i\partial}{\pi\partial\bar{\tau}} - \frac{1}{4\pi\mathrm{Im}\tau}$ is a covariant derivative on modular forms.

This is the final form for the complete string one-loop radiative correction to the appropriate gauge couplings. This result is finite and manifestly invariant under the target space duality group that acts on the T_i, U_i moduli. We see in particular that the (regulated) integrand in our case is related to the partition function of a (3,3) lattice at special values of the (3,3) moduli. The derivative with respect to the R modulus is responsible for the regulation of the IR. In order to see this we will evaluate the part of the radiative correction coming from the low-lying states, which in the unregulated case is responsible for the IR divergence. This is achieved by replacing the (2,2) lattice contribution in eq. (3.35) by 1 and leaving apart for the moment the universal contribution:

$$Z_2^{m=\mu}(Q) = \left[\sum_{i=1}^{3} b_i\right]\int_{\mathcal{F}} \frac{d\tau d\bar{\tau}}{\mathrm{Im}\tau}\mathrm{Im}\tau^{\frac{1}{2}}\left[Z'(R)|_{k+2} - \frac{1}{2}Z'(R)|_{(k+2)/4}\right] \tag{3.37}$$

As expected, $Z_2^{m=\mu}(Q)$ turns out to be finite and for large R behaves like

$$Z_2^{m=\mu}(Q) = (b_1 + b_2 + b_3)[\log(M_{str}^2/Q^2) + 2c_0] + ... \tag{3.38}$$

where the dots stand for terms vanishing in the limit $Q \to 0$. We define M_{str}^2 to be the mass of the lowest lying oscilator state in the string spectrum ($M_{str}^2 = 1/\alpha'$). The constant c_0 can be computed exactly with the result

$$c_0 = \frac{3}{2} - \frac{1}{2}\log(\pi/2) - \frac{1}{2}\psi(1) - \frac{3}{4}\log(3) = 0.738857... \tag{3.39}$$

Observe that the coefficient $b_1 + b_2 + b_3 = 3C(g_a) - T(R_a)$ is nothing but the $N = 1$ β-function coefficient. The constant coefficient c_0, together with that of massive states $F(T_i, U_i)$ as well as the universal contribution define unambiguously the string scheme and can thus be compared with the field theory result (regularized in the IR in the same way as above) in any UV scheme, for instance the conventional $\bar{DR}$. Although this coefficient is small, one has to compute the parts left over including the moduli dependence. In particular the universal contribution can be important. We calculate here the universal contribution due to would be massless states, e.g. the constant part of $\bar{\Omega}/\bar{\eta}^{24}$. This is equal to

$$\frac{60}{\pi}\int_{\mathcal{F}} \frac{d^2\tau}{\mathrm{Im}\tau^2}\sqrt{\mathrm{Im}\tau}\left[Z'(R)|_{k+2} - \frac{1}{2}Z'(R)|_{(k+2)/4}\right] = 20 + \mathcal{O}(1/k) \tag{3.40}$$

This contribute to the coefficient c_0 in (3.38) equal to 1/3 for E_8 and $-5/21$ for E_6. This implies that a full calculation is necessary, namely the contributions from all massive states, in order to find the exact string scheme. The explicit calculation of $Z_2(Q)$ at one-loop, including the moduli dependence, is under way [22].

4 Conclusions

We have presented an IR regularization for string theory (and field theory) induced by the curvature of spacetime as well as by non-trivial dilaton and axion fields. This regularization preserves a form of spacetime supersymmetry and gives masses to all massless fields (including chiral fermions) that are proportional to the curvature.

In the regulated string theory we can compute exactly the one-loop effective action for arbitrarily large, constant, non-abelian magnetic fields. Using this result among other things, we can compute unambiguously the string-induced one-loop threshold corrections to the gauge couplings as functions of the moduli. The eventual integral to be done contains a special subclass of sums associated with (3,3) lattices.

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